Dynamic Games with Incomplete Information

The Perfect Bayesian Equilibrium and its refinements

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Dynamic Bayesian Games

Beyond the classes of games with perfect information and multi-stage games, the SPNE solution concept is of little use because... there are too few subgames

Example: modified entry-game: incumbent has two ways of entering the market that the incumbent is not able to distinguish. In this game there are no proper sub-games so that non credible Nash Equilibria cannot be ruled out.

It would be useful to introduce beliefs about which node a player is playing at. This would allow to apply sequential rationality (in terms of expected payoffs) and to refine the set of possible equilibria of the game. In the example we are in the extreme case as "accommodate" dominates "fight". But, how beliefs should be determined?

Beliefs

Definition

A system of beliefs μ in an extensive form game specifies a probability $\mu(x) \in [0, 1]$ to each decision node x such that

$$\sum_{x \in H} \mu(x) = 1$$

for all information sets H.

This is a subjective assessment of the likelihood of being at any given node x of an information set H

Expected continuation payoff

Let's define $E(u_i|H, \mu, \sigma_i, \sigma_{-i})$ player $i's$ expected payoff conditional on 1) having reached the information set H; 2) the set of beliefs being μ; and 3) the strategies (σ_i, σ_{-i}) which indicate how to continue to play the game. This is the expected continuation payoff.

Example: in the modified entry game, let be $\mu = \frac{1}{2}$, $\sigma_E = in_1$, $\sigma_I = accom$. We have

$$E(u_I|entered, \frac{1}{2}, accom., in_1) = \frac{1}{2}0 + \frac{1}{2}1$$

Sequential Rationality

Definition

A strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ of an extensive form game is sequentially rational at the information set H, given a system of beliefs μ, if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$E(u_{\iota(H)}|H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}) \geqslant E(u_{\iota(H)}|H, \mu, \sigma'_{\iota(H)}, \sigma_{-\iota(H)}) \quad \text{for all } \sigma'_{\iota(H)} \in \Delta S_{\iota(H)}$$

If strategy profile σ satisfies this condition for all information sets, then we say that σ is sequentially rational given the system of beliefs μ.

Practically, σ is sequential rational if at each information set the player who has the move maximizes his continuation payoff, given other players' strategies and the system of beliefs.

Weak Perfect Bayesian Equilibrium

Definition

A strategy profile and a system of beliefs (σ, μ) are a Weak Perfect Bayesian Equilibrium (weak PBE) in an extensive form game if they satisfy:
1) σ is sequentially rational given the system of beliefs μ
2) the system of beliefs is derived from the strategy profile σ by means of the Bayes rule whenever possible. That is, for any information set H reached with a strictly positive probability, $\Pr(H|\sigma) > 0$, we must have

$$\mu(x) = \frac{\Pr(x|\sigma)}{\Pr(H|\sigma)}$$

for all x in H.

Example

Joint Venture between two entrant firms: a firm can decide whether to enter in a market alone, by means of a joint venture with another firm or staying out. If it enters alone it is weak. If it enters in tandem it is strong. The potential partner can accept or refuse the partnership. The incumbent only observes entry, but it cannot distinguish whether the entrant is alone or in tandem. The incumbent must decide whether to "fight" or "accommodate".

Note I

- This definition requires not only that strategies are optimal given the system of beliefs, but that also the system of beliefs is consistent with the equilibrium strategies.
- Important: in the information sets which are never reached by the game path, Bayes rule cannot be applied and beliefs are necessarily arbitrary. There cannot be "reasonable" beliefs (or more "reasonable" than others) in portions of the game that we never observe.
- The attribute "weak" refers to the fact that beliefs outside the equilibrium path remain completely arbitrary, they need to satisfy no constraint, excepting from non negativity and sum to 1.
- Such freedom in determining out-of-equilibrium beliefs is the reason for a large multiplicity of equilibria emerge. Refinements of the weak PBE always include properties that out-of-equilibrium beliefs have to satisfy.

Note II

However even a weak PBE is stronger than a "simple" Nash equilibrium, as the following theorem shows:

Theorem

A strategy profile σ is a Nash equilibrium in the extensive form game if and only if there exists a system of beliefs μ such that
1) the strategy profile σ is sequentially rational given the system of beliefs μ at all information sets H such that $Pr(H|\sigma) > 0$ (that is only at the information sets along the equilibrium path)
2) the system of beliefs is obtained by the strategy profile σ by using the Bayes rule, whenever possible.

Therefore a NE requires sequential rationality only along the equilibrium path and not at all information sets.

Weak PBE and SPNE

Nonetheless, satisfying sequential rationality off the equilibrium path is easy as beliefs are arbitrary. Indeed, the weak PBE is not stronger than SPNE

Example: in an entry game when first the entrant decides whether to enter and next the entrant and the incumbent decide simultaneously whether fighting or accommodating, the only SPNE is $[(in, accom), accom]$ while there exists another weak PBE $[(out, accom), fight]$ with $\mu > \frac{2}{3}$ (to be precise there exist infinite weak PBE, one each belief).

Note: There are various methods one can follow to strengthen the weak PBE solution concept. For instance we can require that (s, μ) produce a weak PBE in each subgame. This implies that a Perfect Bayesian Equilibrium is always also SPNE. In the example before, the weak PBE does not produce a weak PBE in the proper subgame.

Sequential equilibrium (Kreps and Wilson, 1982)

Another refinement of the weak PBE is the concept of sequential equilibrium

Definition

A strategy profile and a system of beliefs (σ, μ) is a sequential equilibrium in an extensive form game Γ_E if it has the following properties:
1) strategy profile σ is sequentially rational given the system of beliefs μ
2) there exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^k = \sigma$ such that $\mu = \lim_{k\to\infty} \mu^k$ where μ^k denotes beliefs derived from the strategy profile σ^k using Bayes rule.

Sequential equilibrium practically requires that beliefs are close to those that would be obtained for small deviations off the equilibrium strategy.
Note: every sequential equilibrium is a weak PBE (but the viceversa does not hold).
Exercise: Apply the concept of SE to the previous examples

Forward induction

Another way to rationalize off-the-equilibrium beliefs is the so called forward induction.
After having observed a deviation, players think that such deviation is rational and desired, not a trembling or a mistake, and that the purpose is that of maximizing the following payoffs.

Example: In the entry game with weak PBE $[out, fight]$ and $\mu(in_1) = 1$, the incumbent should make the following reasoning: "as the entrant did enter, then he certainly did it to play in_2...". Therefore beliefs must be $\mu(in_2) = 1$. But with these off-the-equilibrium beliefs the considered weak PBE is not viable anymore.

But what if the deviation is really a mistake?

Games of signalling and screening

The concept of (weak) PBE can be applied also to dynamic games with asymmetric information (incomplete information). In particular we shall consider two classes of games:
1) signalling games
2) screening games

- In signalling games who has private information is also interested in signalling his type to the non-informed party.
- In screening games the non-informed player wants to induce the informed party to reveal his type.

Signalling games I

Description:

- Nature moves and determines player 1's type, $\theta_1 \in \Theta$
- Types' distribution $F(\theta_1)$ is common knowledge
- Player 1 (sender), after having observed his type, chooses an action $m \in M$ (message)
- Next, observed m, players $2, .., I$ (receivers) simultaneously choose an action $r_i \in R_i$ (answer).
- To player 1, a strategy is a function $a : \Theta \rightarrow M$
- To players $2, ..., I$ a strategy is a function $s_i : M \rightarrow R_i$
- Payoffs are defined over the strategy profiles

Signalling games II

In signalling game we always need to look for

1. Separating equilibria: each "type" sends a different message
2. Pooling equilibria: all "types" send the same message
3. Semi-separating equilibria: senders adopt mixed strategies, so that updating is limited.

If #types>#messages there cannot be separating equilibria

Examples

- the Ph.D. Admission Game
- a simple signalling Game

Refinements I

Off-the-equilibrium beliefs are arbitrary and allow to support many pooling equilibria

How can we get reasonable off-the-equilibrium beliefs in signalling games?

There are several possible refinements of the weak PBE concept to be used in signalling games.
The most simple are the following:
1) passive conjectures: if the probability of making mistakes is independent on player's type, when we are at an information set off-the-equilibrium the probability of facing a given type corresponds to the prior distribution of types

Example: in the Ph.D. Admission Game, $\mu(hater) = 0.9$ at the equilibrium $[(NA; NA), reject]$

Refinements II

2) complete robustness: equilibrium strategies are completely robust if they are best responses whatever the off-the-equilibrium beliefs

Example: in the Ph.D. Admission Game, $[(NA; NA), reject]$ is an equilibrium only if $\mu > \frac{2}{3}$, i.e. not for any possible off-the-equilibrium beliefs.

Refinements III

3) intuitive criterion (Cho, Kreps, 1987) or "equilibrium dominance": if there exist types of the informed player who would be harmed by a deviation off the equilibrium path, whatever the non-informed player beliefs, then the probability to assign to these types off-the-equilibrium is zero (equilibrium strategy dominates over deviations).

Example: in the Ph.D. Admission Game, the equilibrium $[(NA; NA), reject]$ with $\mu > \frac{2}{3}$, does not satisfy the intuitive criterion because the type "hater" never gains by deviating from NA to A whatever university beliefs. Therefore $\mu(hater) = 0$.
Example: in the simple signalling game, t_2 never gains from deviating. Therefore $\mu(t_1) = 1$ and all pooling equilibrium are dropped.

Note

- In separating equilibria, each type sends a different message. If #types = #messages each information set is reached by the equilibrium path. There are no off-the-equilibrium beliefs. Only among pooling equilibria some information sets remain off the equilibrium path and only in this case refinements bite.
- If #messages > #types, also separating equilibria can be refined

Example: 2 types - 3 messages.

Spence (1973) signalling model

- Two types of workers, with high (θ_H) and low (θ_L) innate productivity. The high-type proportion is λ.
- Firms compete on the market and neither cannot distinguish "ability", nor a reliable test exists to measure ability.
- There is a problem of incentives: high-type workers would like to signal their type, while the low-type would like to cheat and declare to be high-type.
- However there exist a signalling device: education. Education does not add anything to human capital
- Before entering the labor market, workers can acquire a level of education observable and verifiable by all players.
- The cost of education is such that

$$
\begin{aligned}
c(0,\theta) &= 0 \quad c_e(e,\theta) > 0 \quad c_{ee}(e,\theta) > 0 \\
c_\theta(e,\theta) &< 0 \quad c_{e\theta}(e,\theta) < 0
\end{aligned}
$$

Intuitively the high-type has an advantage in acquiring education and signalling his type (note: education is costly, it is not cheap talk).

Screening games

In the class of screening games the uninformed player plays first by offering a menu of contracts. The informed player plays next.

The purpose of the uninformed is that of making the informed player revealing his private information.

We still distinguish between separating and pooling equilibrium.

Note: the uninformed player has nothing to signal to the informed player. Therefore a systematic formalization of beliefs is unnecessary.

Note: screening games are dynamic games with incomplete information, but the weak PBE is "degenerate"

Education game - discrete and continuous

Initially Nature determines worker's ability $\alpha = \{2, 5.5\}$ with probability $(1/2, 1/2)$.
Next two firms compete to hire the worker by offering a wage schedule conditional on workers education.
Observed the wage schedule offered, the worker decides his level of education.
Education is costly and costs more to the less able.

discrete version: education $s \in \{0, 1\}$
continuous version: education $s \in [0, 1]$

Important result

No pooling equilibria in screening games where at least one player has discrete strategies